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Unbounded components in the solution sets of strictly quasiconcave vector maximization problems

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Abstract Let (P) denote the vector maximization problem

$$\max\{f(x) = \big(f_1(x), \ldots, f_m(x)\big) \colon x \in D\},$$

where the objective functions f_i are strictly quasiconcave and continuous on the feasible domain D, which is a closed and convex subset of R^n. We prove that if the efficient solution set $E(P)$ of (P) is closed, disconnected, and it has finitely many (connected) components, then all the components are unbounded. A similar fact is also valid for the weakly efficient solution set $E^w(P)$ of (P). Especially, if f_i $(i = 1, \ldots, m)$ are linear fractional functions and D is a polyhedral convex set, then each component of $E^w(P)$ must be unbounded whenever $E^w(P)$ is disconnected. From the results and a result of Choo and Atkins [J. Optim. Theory Appl. **36**, 203–220 (1982)] it follows that the number of components in the efficient solution set of a bicriteria linear fractional vector optimization problem cannot exceed the number of unbounded pseudo-faces of D.

Keywords Strictly quasiconcave vector maximization problem · Efficient solution set · Weakly efficient solution set · Unbounded component · Compactification procedure

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1 Introduction

Consider the *strictly quasiconcave vector maximization problem*

$$(P) \qquad \max\{f(x) = \big(f_1(x), \ldots, f_m(x)\big) : x \in D\},$$

where the objective functions f_i are strictly quasiconcave and continuous on the *feasible domain* D, which is a closed and convex subset of R^n. By definition, a function $\varphi: D \to R$ is said to be *quasiconcave* on D if for any $x^1,\ x^2 \in D$ and for any $t \in (0,1)$ it holds

$$\varphi(tx^1 + (1-t)x^2) \geq \min\{\varphi(x^1), \varphi(x^2)\}.$$

We say that φ is *strictly quasiconcave* on D, if φ is quasiconcave on D and for any $x^1,\ x^2 \in D$ satisfying $\varphi(x^1) \neq \varphi(x^2)$, and for any $t \in (0,1)$, it holds

$$\varphi(tx^1 + (1-t)x^2) > \min\{\varphi(x^1), \varphi(x^2)\}.$$

If $x \in D$ and one cannot find any $y \in D$ such that $f(x) \leq f(y)$ and $f(y) \neq f(x)$, then x is said to be an *efficient solution* of (P). If $x \in D$ and there does not exist $y \in D$ such that $f(x) < f(y)$, then x is called a *weakly efficient solution* of (P). As usual, for any $w, w' \in R^m$, the inequality $w \leq w'$ (resp., $w < w'$) means $w_i \leq w'_i$ (resp., $w_i < w'_i$) for all $i = 1, \ldots, m$.

It is easy to verify that if $\varphi: R^n \to R$ is a linear fractional function defined on D (i.e. $\varphi(x) = (a^T x + \alpha)/(b^T x + \beta)$ for some $a, b \in R^n, \alpha, \beta \in R$, and $b^T x + \beta \neq 0$ for all $x \in D$), then φ is strictly quasiconcave on D.

As a special case of (P), we consider the following *linear fractional vector optimization problem* (LFVO problem, for brevity):

$$(P_1) \qquad \max\{f(x) = (f_1(x), \ldots, f_m(x)) : x \in R^n,\ Cx \geq d\},$$

where C is an $(r \times n)$-matrix, d is an r-dimensional column vector, and

$$f_i(x) = (a_i^T x + \alpha_i)/(b_i^T x + \beta_i) \quad (i = 1, \ldots, m)$$

are linear fractional functions defined on the set $D = \{x \in R^n : Cx \geq d\}$.

The efficient solution set and the weakly efficient solution set of (P) (resp., of (P_1)) are denoted by $E(P)$ and $E^w(P)$) (resp., by $E(P_1)$ and $E^w(P_1)$).

We now introduce some standard notions and notation. A subset Z of an Euclidean space is said to be *connected* if one cannot find any pair (Z_1, Z_2) of disjoint nonempty open subsets Z_1, Z_2 of Z in the induced topology such that $Z = Z_1 \cup Z_2$. One says that Z is *path connected* if for any $a, b \in Z$ there exists a continuous mapping $\gamma: [0,1] \to Z$ such that $\gamma(0) = a$, $\gamma(1) = b$. If for any given points $a, b \in Z$ there exists a sequence of line segments $[z_i, z_{i+1}] \subset Z$ $(i = 0, \ldots, k-1)$ such that $z_0 = a$ and $z_k = b$, then Z is said to be *connected by line segments*. If Z is disconnected, then we denote by $\chi(Z)$ *the (cardinal) number of components* of Z. By definition, a subset $M \subset Z$ is said to be a *component* of Z if M is connected and it is not a proper subset of any connected subset of Z. The closed ball with the center at x and radius $\varepsilon > 0$ is denoted by $\bar{B}(x, \varepsilon)$, while its interior is denoted by $B(x, \varepsilon)$. For a subset $Z \subset R^k$ and a constant $\rho > 0$, we put

$$\mathrm{dist}(x, Z) := \inf\{\|x - u\| : u \in Z\}$$

for all $x \in R^k$, and $B(Z, \rho) := \{x \in R^k : \mathrm{dist}(x, Z) < \rho\}$.

Topological properties of the solution sets of strictly quasiconvex vector minimization problems have been discussed by several authors (see Schaible 1983; Warburton 1983; Luc 1987, 1989; Daniilidis et al. 1997; Benoist 1998; Huy and Yen 2004, 2005, and the references therein). Apart from Huy and Yen (2004, 2005), the feasible domain is assumed to be compact.

The following result of Benoist (1998) extends the preceding results in Schaible (1983) and in Daniilidis et al. (1997) where the cases $m = 2$ and $m = 3$ were treated.

Theorem 1.1 *If D is bounded, then $E(P)$ is connected. Thus, if the feasible domain of (P_1) is bounded, then $E(P_1)$ is connected.*

The connectedness of $E(P_1)$ in the case where D is a bounded set can be proved by using several known results on monotone affine variational inequalities (see Yen and Phuong 2000).

The next theorem is due to Choo and Atkins (1983).

Theorem 1.2 *If the feasible domain of (P_1) is bounded, then $E^w(P_1)$ is connected by line segments.*

Using Theorem 1.2, the second assertion of Theorem 1.1, and a compactification procedure, Phu (1998, Private Communication) obtained the following result.

Theorem 1.3 *If the sets $E(P_1)$ and $E^w(P_1)$ are bounded, then they are connected.*

Note that the efficient solution set and the weakly efficient solution set of (P) may be disconnected if D is unbounded (see Choo and Atkins 1983). In Hoa et al. (2005a) it was proved that for any integer m there exist LFVO problems with m objective criteria whose efficient solution set and weakly efficient solution set have exactly m components.

The aim of this paper is to extend Theorem 1.3 to the case of strictly quasiconcave vector maximization problems and prove that, under some additional conditions, if the efficient solution set (or the weakly efficient set) of (P) is disconnected, then all the components in the set are unbounded. The facts help us to understand better the topological structure of the solution sets of strictly quasiconcave vector maximization problems with unbounded feasible domains. As an application, we give a rough estimate for the number of components in the efficient solution set of a bicriteria linear fractional vector optimization problem.

The main results are obtained in Sects. 2 and 3. An application of the results to bicriteria LFVO problems is given in Sect. 4.

2 Extension of Theorem 1.3

In this section, we will extend Theorem 1.3 to the case of strictly quasiconcave vector maximization problems.

The next simple lemma shows that, for a feasible point, the property of being an efficient solution or a weakly efficient solution of (P) has a local character.

Lemma 2.1

(a) *If $x \in D$ is a local efficient solution of (P), that is there exists $\varepsilon > 0$ such that x is an efficient point of the problem*

$$\max\{f(z)\colon z \in D \cap \bar{B}(x, \varepsilon)\}, \tag{2.1}$$

then $x \in E(P)$.

(b) *If* $x \in D$ *is a local weakly efficient solution of* (P)*, that is there exists* $\varepsilon > 0$ *such that* x *is a weakly efficient point of the problem* (2.1)*, then* $x \in E^w(P)$.

Proof We will only prove (a). The proof of (b) is similar. Suppose that x, which is the center of the ball $\bar{B}(x,\varepsilon)$, is an efficient solution of (2.1). If $x \notin E(P)$, then there exist $y \in D$ and $i_0 \in \{1,\ldots,m\}$ such that $f(x) \leq f(y)$ and $f_{i_0}(x) < f_{i_0}(y)$. Choose $t \in (0,1)$ as small as $y_t := (1-t)x + ty$ belongs to $D \cap \bar{B}(x,\varepsilon)$. From the quasiconcavity of f_i it follows that

$$f_i(y_t) \geq \min\{f_i(x), f_i(y)\} = f_i(x)$$

for all $i = 1,\ldots,m$. The strict quasiconcavity of f_{i_0} and the inequality $f_{i_0}(x) < f_{i_0}(y)$ imply $f_{i_0}(x) < f_{i_0}(y_t)$. This contradicts the assumption that x is an efficient solution of (2.1). □

Theorem 2.1 *If the efficient solution set* $E(P)$ *is bounded, then it is connected.*

Proof To obtain a contradiction, suppose that $E(P)$ is disconnected. Then there exist open subsets U and V of R^n such that

$$U \cap E(P) \neq \emptyset, \quad V \cap E(P) \neq \emptyset, \quad \Big(U \cap E(P)\Big) \cap \Big(V \cap E(P)\Big) = \emptyset, \quad E(P) \subset U \cup V. \tag{2.2}$$

Since $E(P)$ is bounded, there is $\rho > 0$ such that $E(P) \subset B(0,\rho)$. By Theorem 1.1, the efficient solution set of the vector maximization problem

$$\max\{f(x) : x \in D \cap \bar{B}(0,2\rho)\}, \tag{2.3}$$

which is abbreviated to E_0, is connected. Let $U_0 = U \cap B(0,\rho)$, $V_0 = V \cup \Big(R^n \setminus \bar{B}(0,\rho)\Big)$. We now show that

$$U_0 \cap E_0 \neq \emptyset, \qquad V_0 \cap E_0 \neq \emptyset, \tag{2.4}$$

$$\Big(U_0 \cap E_0\Big) \cap \Big(V_0 \cap E_0\Big) = \emptyset, \tag{2.5}$$

$$E_0 \subset U_0 \cup V_0. \tag{2.6}$$

From Lemma 2.1, it follows that $E(P) \cap B(0,2\rho) = E_0 \cap B(0,2\rho)$. Combining this with the inclusion $E(P) \subset B(0,\rho)$ we deduce that $E(P) = E_0 \cap B(0,2\rho)$, hence (2.4) is a consequence of the first two properties in (2.2). Since $U_0 = U \cap B(0,\rho)$ and $V_0 = V \cup \Big(R^n \setminus \bar{B}(0,\rho)\Big)$, we have

$$\Big(U_0 \cap E_0\Big) \cap \Big(V_0 \cap E_0\Big) \subset \Big(U \cap E(P)\Big) \cap \Big(V \cap E(P)\Big).$$

Thus, the third property in (2.2) implies (2.5). Finally, the inclusion (2.6) follows from the fourth property in (2.2) and the inclusion $E_0 \subset E(P) \cup \Big(R^n \setminus \bar{B}(0,\rho)\Big)$. We have arrived at a contradiction, because (2.4)–(2.6) imply that E_0 is a disconnected set. □

In order to extend the assertion of Theorem 1.3 related to the weakly efficient solution set to the case of strictly quasiconcave vector maximization problems, we need a lemma, which is a special case of Theorem 4.1 in Warburton (1983). The proof given below shows that the lemma can be seen also as a corollary of Theorem 1.1.

Lemma 2.2 *If D is bounded, then $E^w(P)$ is connected.*

Proof There is no loss of generality in assuming that D is nonempty. Since the set $E(P)$ is nonempty (see Luc 1989) and connected by Theorem 1.1, it suffices to show that "*for each $u^0 \in E^w(P)$ there exists a sequence $\{u^1, u^2, \ldots, u^m\} \subset D$ such that the line segments $[u^{i-1}, u^i]$ $(i = 1, \ldots, m)$ are contained in $E^w(P)$, and $u^m \in E(P)$*".

Let u^1 be a solution of the optimization problem

$$\max\{f_1(z): z \in D,\ f(z) \geq f(u^0)\}. \tag{2.7$_1$}$$

For any $i \in \{2, \ldots, m\}$, if u^{i-1} has been chosen, then as u^i we choose an arbitrary solution of the optimization problem

$$\max\{f_i(z): z \in D,\ f(z) \geq f(u^{i-1})\}. \tag{2.7$_i$}$$

(By the compactness of D and the continuity of f_i, problem $(2.7)_i$ has at least one solution.) Using the quasiconcavity of the functions f_i, by induction it is easy to verify that the line segments $[u^{i-1}, u^i]$ $(i = 1, \ldots, m)$ are contained in $E^w(P)$. If $u^m \notin E(P)$, then there exist $y \in D$ and $i_0 \in \{1, \ldots, m\}$ such that $f(u^m) \leq f(y)$ and $f_{i_0}(u^m) < f_{i_0}(y)$. Since $f(u^{i_0-1}) \leq f(u^m)$ and $f(u^{i_0}) \leq f(u^m)$, this implies that u^{i_0} cannot be a solution of $(2.7)_{i_0}$, a contradiction. Thus $u^m \in E(P)$. □

Theorem 2.2 *If $E^w(P)$ is bounded, then it is connected.*

Proof It suffices to apply Lemmas 2.1, 2.2, and arguments similar to those in the proof of Theorem 2.1. □

3 Unboundedness of the components in disconnected solution sets

It turns out that the set $E^w(P_1)$ cannot have any bounded component if it is disconnected. A similar property also holds for the sets $E(P_1)$, $E(P)$, and $E^w(P)$ under certain mild assumptions.

Theorem 3.1 *The following properties hold:*

(1) *If $E^w(P_1)$ is disconnected, then each component in $E^w(P_1)$ is unbounded.*
(2) *If $E(P_1)$ is disconnected, closed, and $\chi(E(P_1))$ is finite, then each component in $E(P_1)$ is unbounded.*

Proof

(1) The set $E^w(P_1)$ is closed (see Choo and Atkins 1983). Since the components of $E^w(P_1)$ are closed in the induced topology, they are closed subsets of R^n. Suppose the first assertion of the theorem is false. Then $E^w(P_1)$ has a bounded component M_0 and there is a point $\widehat{x} \in E^w(P_1)\backslash M_0$. Fix a point $\bar{x} \in M_0$. Let $\mu_0 = \max\{\|u\|: u \in M_0\}$,

$$\rho = \max\left\{|\widehat{x}_1| + 1, \ldots, |\widehat{x}_n| + 1, \mu_0 + 1\right\}$$

and $\Pi = [-\rho, \rho] \times \cdots \times [-\rho, \rho]$. It is clear that

$$M_0 \subset \operatorname{int} \Pi = (-\rho, \rho) \times \cdots \times (-\rho, \rho).$$

Consider the LFVO problem

$$(P_1') \qquad \min\{f(x) : x \in D_\Pi\},$$

where

$$D_\Pi = D \cap \Pi = \{x : Cx \geq d,\ -\rho \leq x_i \leq \rho,\ i = 1, \ldots, n\}.$$

Since $E^w(P_1) \cap \Pi \subset E^w(P_1')$, we have $\{\bar{x}, \widehat{x}\} \subset E^w(P_1')$. As D_Π is compact, $E^w(P_1')$ is connected by line segments (see Theorem 1.2). Then there exist $x^1, x^2, \ldots, x^k$ in $E^w(P_1')$ such that $x^1 = \bar{x}$, $x^k = \widehat{x}$, and

$$[x^i, x^{i+1}] \subset E^w(P_1') \quad \text{for} \quad i = 1, \ldots, k-1.$$

It is clear that there must exists an index $j \in \{1, \ldots, k-1\}$ such that $x^j \in M_0$, but $x^{j+1} \notin M_0$. On one hand, since Π is convex and $x^j \in \operatorname{int} \Pi$,

$$[x^j, x^{j+1}) := \{(1-t)x^j + tx^{j+1} : 0 \leq t < 1\} \subset \operatorname{int} \Pi.$$

On the other hand, since $E^w(P_1') \cap \operatorname{int} \Pi \subset E^w(P_1)$ by Lemma 2.1, we have $[x^j, x^{j+1}) \subset E^w(P_1)$. As $x^j \in M_0$ and M_0 is a component of $E^w(P_1)$, this implies $[x^j, x^{j+1}) \subset M_0$. From the closedness of M_0 it follows that $[x^j, x^{j+1}] \subset M_0$. This contradicts the fact that $x^{j+1} \notin M_0$.

(2) This assertion follows from the second assertion in the next theorem. □

Theorem 3.2 *The following properties hold:*

1. *If $E^w(P)$ is disconnected and $\chi(E^w(P))$ is finite, then each component in $E^w(P)$ is unbounded.*
2. *If $E(P)$ is disconnected, closed, and $\chi(E(P))$ is finite, then each component in $E(P)$ is unbounded.*

Proof Since the proofs of (1) and (2) are similar, we will only prove (2). From the assumptions it follows that every component of $E(P)$ is a closed subset of R^n. To obtain a contradiction, suppose that $E(P)$ is disconnected and it has a bounded component M_0. Then M_0 is a compact set. Let

$$E(P) \backslash M_0 = \bigcup_{i=1}^{\ell} M_i,$$

where M_i, $i = 1, \ldots, \ell$, are different components of $E(P)$. We put $K = \bigcup_{i=1}^{\ell} M_i$. Of course, K is nonempty and closed. Let

$$\rho_0 = \inf_{x \in M_0} \operatorname{dist}(x, K).$$

It is well known that $\operatorname{dist}(\cdot, K)$ is a Lipschitz function on R^n with the Lipschitz constant 1. Since $\operatorname{dist}(x, K) > 0$ for all $x \in M_0$ and M_0 is compact, we have $\rho_0 > 0$. Setting $U = B\big(M_0, \frac{\rho_0}{2}\big)$, $V_0 = B\big(K, \frac{\rho_0}{2}\big)$, we see that U, V_0 are disjoint open sets satisfying

$$U \neq \emptyset, \quad V_0 \neq \emptyset, \quad E(P) \subset U \cup V_0.$$

Fix a point $\widehat{x} \in K$. Let $\mu_0 = \max\{\|u\| : u \in M_0\}$,

$$\rho = \max\{|\widehat{x}_1| + 1, \ldots, |\widehat{x}_n| + 1, \mu_0 + \rho_0 + 1\}.$$

Consider the box $\Pi = [-\rho, \rho] \times \cdots \times [-\rho, \rho]$ and the vector optimization problem

$$(P') \qquad \min\{f(x) : x \in D_\Pi\},$$

where $D_\Pi = D \cap \Pi$. We have $\operatorname{dist}(x, \partial\Pi) > \rho_0$ for all $x \in M_0$, where $\partial\Pi$ denotes the boundary of Π. Put

$$V = \big(V_0 \cap \operatorname{int}\Pi\big) \cup B\big(\partial\Pi, \frac{\rho_0}{2}\big).$$

From Lemma 2.1 and the construction of U, V it follows that $U \neq \emptyset, V \neq \emptyset, U \cap V = \emptyset$, and

$$U \cup V \supset E(P').$$

Hence $E(P')$ is disconnected. We have arrived at a contradiction, because $E(P')$ is connected by Theorem 1.1. □

It would be desirable to remove the assumptions on the closedness of the set $E(P_1)$ (resp., $E(P)$) and the finiteness of the number $\chi(E(P_1))$ (resp., $\chi(E(P))$) from the second assertion of Theorem 3.1 (resp., Theorem 3.2). Also, it would be nice if the assumption on the finiteness of the number $\chi(E^w(P))$ in the first assertion of Theorem 3.2 can be omitted. In our opinion, *proving these general statements by using Theorem 1.1 and a compactification procedure, in which* (P_1) *and* (P) *are replaced by vector optimization problems with compact feasible domains, is not an easy task.*

The next example is designed to clarify the above remark.

Example 3.1 Let M be a subset of R^2 defined by setting

$$M = ([0,1] \times \{0\}) \cup \left(\bigcup_{k=1}^{\infty} \left([0, +\infty) \times \left\{ \frac{1}{k} \right\} \right) \right).$$

Let

$$M_0 = [0,1] \times \{0\}, \quad \Pi = [-2,2] \times [-2,2], \quad \widetilde{M} = (M \cap \Pi) \cup (\{2\} \times [0,1]).$$

(Here M and $\widetilde{M}$ resemble, respectively, the sets $E(P)$ and E_0 in the proof of Theorem 2.1 with the ball $\bar{B}(0, 2\rho)$ being replaced by the box Π.) Note that $\widetilde{M}$ is the union of $M \cap \Pi$ and one part of the boundary of Π. Note also that $\widetilde{M} \cap \operatorname{int}\Pi = M \cap \operatorname{int}\Pi$. We now show that:

(1) M_0 is a bounded component of M;
(2) $\widetilde{M}$ is connected.

Therefore, *after adding to* $M \cap \Pi$ *some boundary points of* Π *to have* $\widetilde{M}$, M_0 *is no longer a component of the new set!* To prove (1), suppose to the contrary that there exists a connected set $M_1 \subset M$ such that $M_0 \subset M_1$ and $M_1 \neq M_0$. Fix a point $\bar{x} \in M_1 \setminus M_0$. There is a unique integer $l \geq 2$ such that $\bar{x} \in [0, +\infty) \times \left\{\frac{1}{l}\right\}$. Put

$$U = \left\{ x = (x_1, x_2) \in R^2 : x_2 < \frac{1}{2}\left(\frac{1}{l} + \frac{1}{l+1}\right) \right\},$$

$$V = \left\{ x = (x_1, x_2) \in R^2 : x_2 > \frac{1}{2}\left(\frac{1}{l} + \frac{1}{l+1}\right) \right\}.$$

It is clear that

$$U \cap M_1 \neq \emptyset, \quad V \cap M_1 \neq \emptyset, \quad M_1 \subset U \cup V, \quad U \cap V = \emptyset.$$

This shows that M_1 is disconnected, a contradiction. To prove (2), suppose to the contrary that there exist open subsets $U,\ V$ of R^2 such that

$$U \cap \widetilde{M} \neq \emptyset, \quad V \cap \widetilde{M} \neq \emptyset, \quad \widetilde{M} \subset [U \cap \widetilde{M}] \cup [V \cap \widetilde{M}], \quad [U \cap \widetilde{M}] \cap [V \cap \widetilde{M}] = \emptyset.$$

Since M_0 is connected, we may assume that $M_0 \subset U$. As U is an open set, for l large enough we have $\left([0,+\infty) \times \left\{\frac{1}{l}\right\}\right) \cap U \neq \emptyset$. Hence

$$\left([0,+\infty) \times \left\{\frac{1}{l}\right\}\right) \cup (\{2\} \times [0,1]) \subset U,$$

because the set on the left-hand side of the last inclusion is path connected. Then it is easy to see that $\widetilde{M} \subset U$. But this implies that $V \cap \widetilde{M} = \emptyset$, which is impossible.

4 An application

In this section, we show that Theorem 3.1 leads to a rough estimate for the number $\chi(E(P_1))$.

Let C_j^T denote the column vector corresponding to the jth row of the matrix $C \in R^{r\times n}$. Let d_j denote the jth component of the vector $d \in R^r$. A *pseudo-face* of the polyhedral convex set

$$D = \{x \in R^n : Cx \geq d\}$$

is the set D_α of all $x \in R^n$ satisfying the system

$$C_j^T x = d_j \quad \text{for } i \in \alpha, \qquad C_j^T x > d_j \quad \text{for } i \notin \alpha,$$

where $\alpha \subset \{1,2,\ldots,r\}$ is a set of indexes. It is clear that the number of pseudo-faces of D cannot exceed 2^r.

Proprsition 4.1 *For any bicriteria linear fractional vector optimization problem of the form* (P_1), *if* $E(P_1)$ *is a disconnected closed set, then* $\chi(E(P_1))$ *cannot exceed the number of unbounded pseudo-faces of* D.

Proof According to Corollary 3.1 from Choo and Atkins (1982), the intersection of the efficient solution set of (P_1) with a pseudo-face D_α of D is a convex set. So, D_α can have nonempty intersection with not more than one component of $E(P_1)$. In particular, $\chi(E(P_1))$ is less or equal the number of pseudo-faces of D. Since $E(P_1)$ is a disconnected closed set, each component in $E(P_1)$ is unbounded by Theorem 3.1. It follows that $\chi(E(P_1))$ cannot exceed the number of unbounded pseudo-faces of D. □

Let us consider two illustrative examples.

Example 4.1 (see Hoa et al. 2005a) Let $n = m = 2$,

$$D = \left\{x = (x_1,x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0,\ x_1 + x_2 \geq 1\right\}$$

and

$$f_i(x) = \frac{-x_i + \frac{1}{2}}{x_1 + x_2 - \frac{3}{4}} \qquad (i = 1, 2).$$

In this case, we have

$$E(P_1) = E^w(P_1) = \big([1,\infty) \times \{0\}\big) \cup \big(\{0\} \times [1,\infty)\big), \quad \chi(E(P_1)) = \chi(E^w(P_1)) = 2,$$

while the number of unbounded pseudo-faces of D is 3.

Example 4.2 (see Hoa et al. 2005b) Let $n = m = 2$,

$$D = \{x \in R^2 : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \geq 1\},$$

$$f_1(x) = \frac{x_1 + 1}{2x_1 + x_2}, \quad f_2(x) = \frac{-x_1 - 2}{x_1 + x_2}.$$

In this case, we have

$$E(P_1) = E^w(P_1) = \Big([1,+\infty) \times \{0\}\Big) \cup \Big([0,+\infty) \times \{2\}\Big) \cup \Big(\{0\} \times [2,+\infty)\Big).$$

Hence $\chi(E(P_1)) = \chi(E^w(P_1)) = 2$, while the number of unbounded pseudo-faces of D is 3. It is worthy to stress that the efficient solution set has nonempty intersections with all the three unbounded pseudo-faces. (There are two unbounded pseudo-faces of D which have nonempty intersections with one component of $E(P_1)$.)

One may conjecture that if $m = 2$ then

$$\chi(E(P_1)) \leq 2, \quad \chi(E^w(P_1)) \leq 2, \quad \chi(E(P)) \leq 2, \quad \chi(E^w(P)) \leq 2.$$

More generally,

$$\max\{\chi(E(P_1)), \chi(E^w(P_1))\} \leq \min\{m, n\}$$

and

$$\max\{\chi(E(P)), \chi(E^w(P))\} \leq \min\{m, n\}.$$

None of these six estimates has been established so far.

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